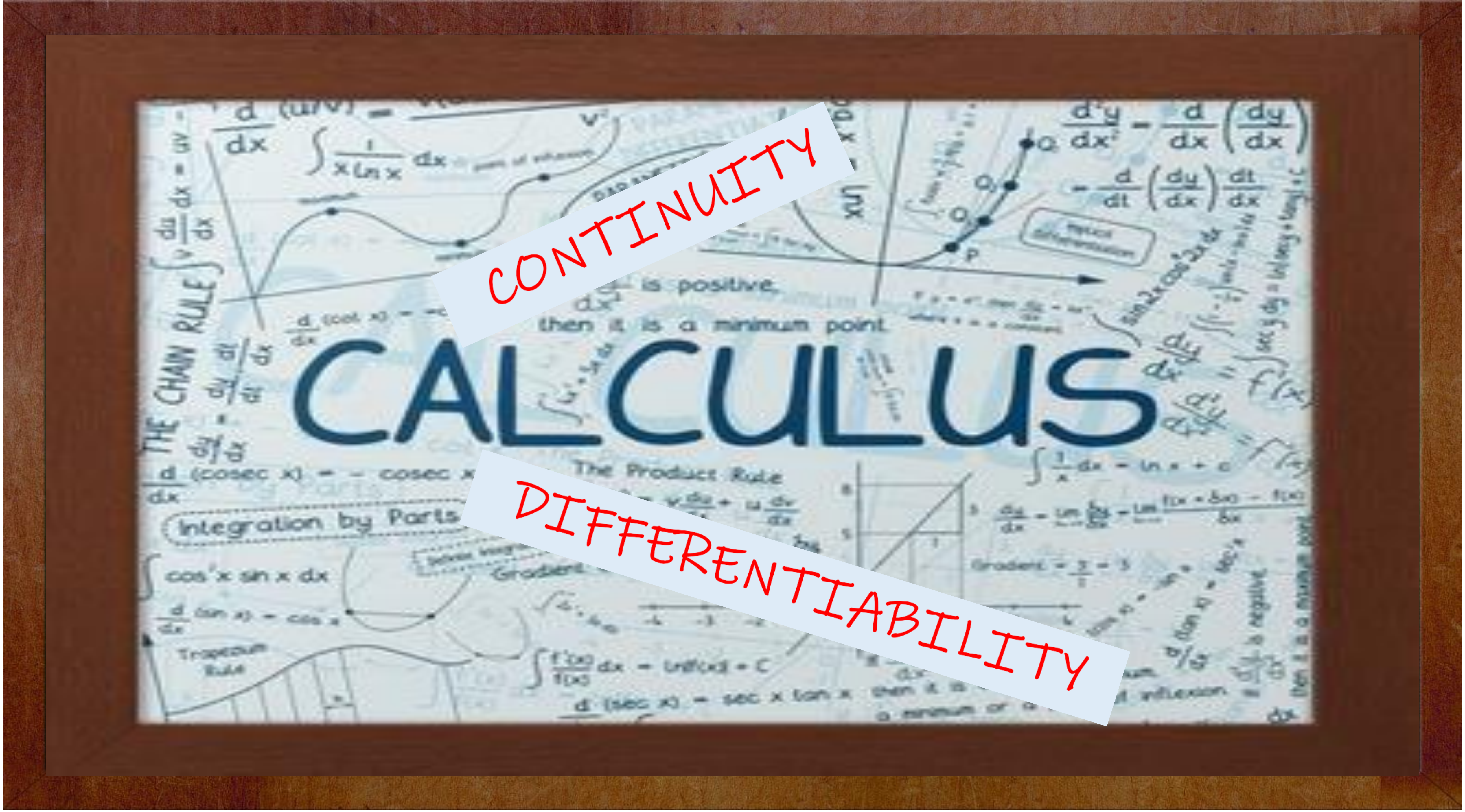

CONTINUITY
CALCULUS
DIFFERENTIABILITY
THE CHAIN RULE
Integration by Parts
The Product Rule
then it is a minimum point

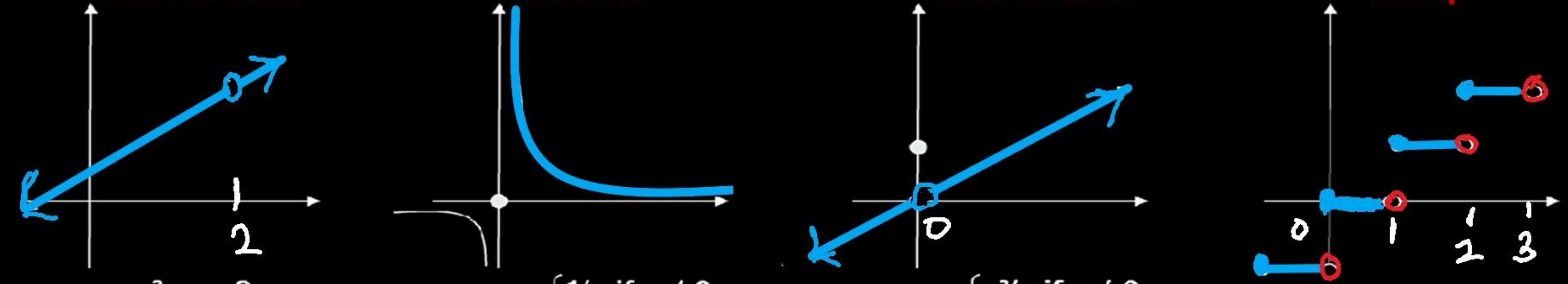
Continuity

Definition: A function is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$

Note: that the definition implicitly requires three things of the function

1. f(a) is defined (i.e., a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exisits
3. $\lim_{x \to a} f(x) = f(a)$

f has a discontinuity at a, if f is not continuous at a. Note the graphs of the examples of discontinuities below:



Removable: $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

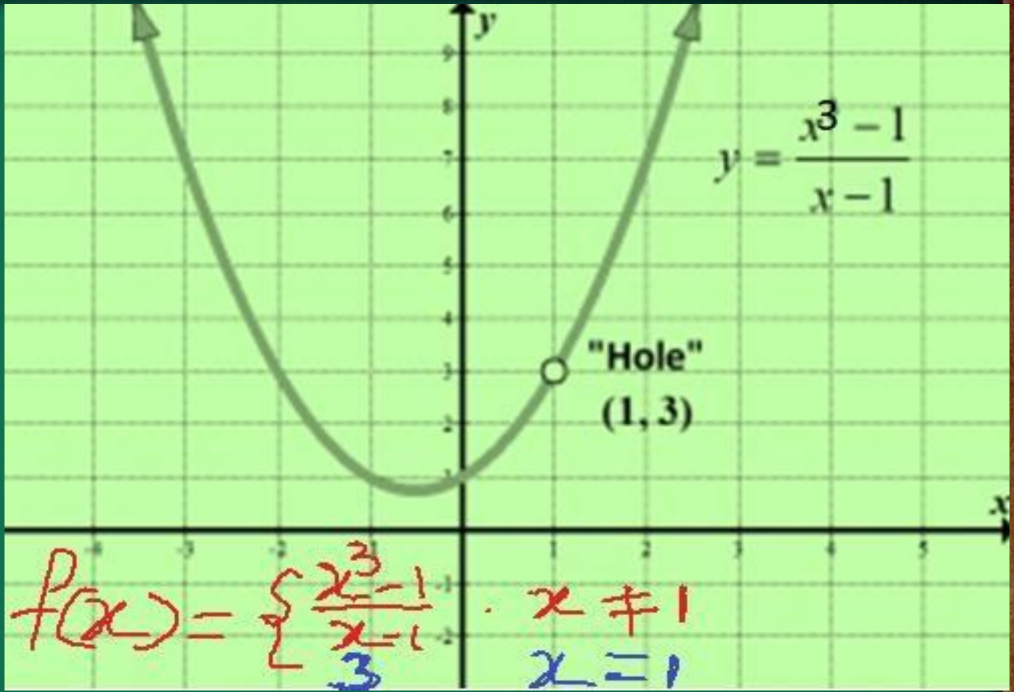
Infinite: $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Removable: $f(x) = \begin{cases} x^2/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Jump: $f(x) = [[x]]$

Types of Discontinuities…

- **Removable Discontinuities** – <u>can</u> be "repaired"
  - *Hole* (factor can be "factored out" of the denominator)
- **Essential Discontinuities** – <u>cannot</u> be "repaired"
  - *Jumps* (usually found in piecewise functions)
  - *Asymptotes* (can't remove a factor/problem in the denominator) --- (like 1/x)



$$\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1}$$

$$= (x^2 + x + 1)$$

$$= (x + \frac{1}{2})^2 + \frac{3}{4}$$

- ***Continuity*** *– no gaps in the curve (layman's definition)*
- ***Discontinuity*** *– a point where the function is not continuous*
- ***Removable discontinuity*** *– a discontinuity that can be removed by redefining the function at a point also called a point discontinuity*
- ***Infinite discontinuity*** *– a discontinuity because the function increases or decreases without bound at a point*
- ***Jump discontinuity*** *– a discontinuity because the function jumps from one value to another*
- ***Continuous from the right at a number a*** *– the limit of f(x) as x approaches a from the right is f(a)*
- ***Continuous from the left at a number a*** *– the limit of f(x) as x approaches a from the left is f(a)*

- *A function is continuous on an interval if it is continuous at every number in the interval*

Limit Laws

If $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = N$

Sum $\lim_{x\to a}[f(x)+g(x)] = M+N$

Difference $\lim_{x\to a}[f(x)-g(x)] = M-N$

Constant $\lim_{x\to a}[k\cdot f(x)] = k\cdot M$

Product $\lim_{x\to a}[f(x)\,g(x)] = M\cdot N$

Quotient $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{M}{N}$ $N \neq 0$

Power $\lim_{x\to a}[f(x)]^n = M^n$ n is a positive integer

Root $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{M}$ n is a positive integer

Limit Formulas:

1. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

2. $\lim_{x\to 0}\frac{\sin^{-1} x}{x} = \lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$

3. $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

4. $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$

5. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

6. $\lim_{x\to a}\frac{x^n-a^n}{(x-a)} = n.a^{n-1}$

QUESTIONS ON CONTINUITY

1. Let $f(x) = \begin{cases} \cos(x) + 1 & , \text{if } x \leq 0; \\ 2 - 3x & , \text{if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

Solution:

1. The function is defined at $x = 0$ and the value is $f(0) = \cos(0) + 1 = 2$.
2. Since $y = \cos(x) + 1$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \cos(x) + 1 = \cos(0) + 1 = 2.$$

3. Since $y = 2 - 3x$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2 - 3x = 2 - 3(0) = 2.$$

Since all three of these values are the same, the function is continuous at $x = 0$.

Q 2. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & , \text{if } x \neq 0; \\ 1 & , \text{if } x = 0. \end{cases}$. Is f continuous at $x = 0$?

Solution:

1. The function is defined at $x = 0$ and its value is $f(0) = 1$.
2. Now we use the squeeze theorem to find the value of the limit.

Since $-1 \leq \sin\left(t\frac{1}{x}\right) \leq 1$ for all values of x, we can multiply by x^2 to get $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all values of x. Since $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$, we conclude that the function between them also approaches zero. Therefore $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, which implies $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) + 3 = 3$.

Since the value of limit does NOT equal the value of the function, $f(x)$ is NOT continuous at $x = 0$.

3. Let $f(x)=\begin{cases}\frac{x^2-9}{x-3} & \text{, if } x<3;\\ cx^2+10 & \text{, if } x\ge 3.\end{cases}$ Find the value of c so that $f(x)$ is continuous at $x=3$

Solution:

1. The function is defined at $x=3$ and its value is $f(3)=c(3)^2+10=9c+10$.

2. $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3^-}\dfrac{(x-3)(x+3)}{x-3}=6.$

3. Since $y=cx^2+10$ is continuous at $x=3$, we have:

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} cx^2+10=9c+10.$$

In order to make all three of these the same, we need $9c+10=6$. Thus, $c=-\frac{4}{9}$.

4. **Find all points of discontinuity of f, where f is defined by** $f(x)=\begin{cases}\dfrac{|x|}{x}, & if \quad x\neq 0\\ 0, & if \quad x=0\end{cases}$

Ans.

$$LHL=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{|x|}{x}$$

Putting $x=0-h$ as $x\longrightarrow 0^-$ when $h\longrightarrow 0$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{h\to 0}\frac{|0-h|}{0-h}$$

$$=\lim_{h\to 0}\frac{h}{-h}=-1$$

$$RHL=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{|x|}{x}$$

Putting $x=0+h$ as $x\longrightarrow 0^+$; $h\longrightarrow 0$

$$\therefore \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{|0+h|}{0+h}$$

$$=\lim_{h\to 0}\frac{h}{h}=1$$

$\therefore$ LHL $\neq$ RHL. Thus, $f(x)$ is discontinuous at $x=0$.

5. Let $G(x) = \begin{cases} \frac{1}{(x+3)^2} & \text{, if } x \leq -1; \\ 2-x & \text{, if } -1 < x \leq 1; \\ \frac{3}{x+2} & \text{, if } x > 1. \end{cases}$. Find all values of x where G is not continuous.

Solution: There are four points to immediately consider: $x = -3$ and $x = -2$ because they make a denominator zero as well as $x = -1$ and $x = 1$ because the function rule changes at these values.

x = −3: Since $y = \frac{1}{(x+3)^2}$ is discontinuous at $x = -3$ and $G(x)$ uses this rule for $x < -1$, we see that $G(x)$ is NOT continuous at $x = -3$.

x = −2: Even through $y = \frac{3}{x+2}$ is discontinuous at $x = -2$, the function $G(x)$ only uses the rule $y = \frac{3}{x+2}$ for values where $x > 1$ and the rule it does use at $x = -2$ is continuous at that value. So $G(x)$ is continuous at $x = -2$.

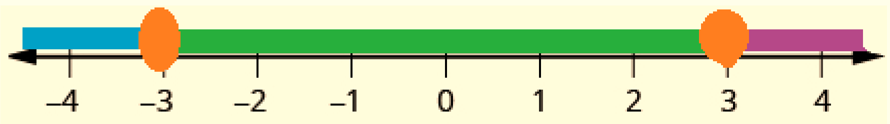
x = −1: $\lim_{x \to -1^-} G(x) = \dfrac{1}{(-1+3)^2} = \dfrac{1}{4}$ and $\lim_{x \to -1^+} G(x) = 2 - (-1) = 3$. Since these are not the same, the function $G(x)$ is NOT continuous at $x = -1$.

x = 1: $\lim_{x \to 1^-} G(x) = 2 - (1) = 1$ and $\lim_{x \to 1^+} G(x) = \dfrac{3}{1+3} = 1$. Since these ARE the same and they equal the value of the function at $x = 1$, the function $G(x)$ is continuous at $x = 1$.

Therefore, the function $G(x)$ is continuous everywhere except $x = -3$ and $x = -1$.

Question 7: **Discuss the continuity of the following function**

$$f(x)=\begin{cases}|x|+3, & \text{If } x\le -3\\ -2x, & \text{If } -3<x<3\\ 6x+2, & \text{If } x\ge 3\end{cases}$$



Answer 7:

Let, k be any real number. According to question, $k<-3$ or $k=-3$ or $-3<k<3$ or $k=3$ or $k>3$

First case: If, $k<-3$,

$f(k)=-k+3$ and $\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k}(-x+3)=-k+3$. Here, $\lim\limits_{x\to k} f(x)=f(k)$

Hence, the function f is continuous for all real numbers less than -3.

Second case: If, $k=-3$, $f(-3)=-(-3)+3=6$

LHL = $\lim\limits_{x\to -3^-} f(x)=\lim\limits_{x\to -3^-}(-x+3)=-(-3)+3=6$

RHL = $\lim\limits_{x\to -3^+} f(x)=\lim\limits_{x\to -3^+}(-2x)=-2(-3)=6$. Here, $\lim\limits_{x\to k} f(x)=f(k)$

Hence, the function f is continuous at $x=-3$.

Third case If, $-3<k<3$,

$f(k)=-2k$ and $\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k}(-2x)=-2k$. Here, $\lim\limits_{x\to k} f(x)=f(k)$

Hence, the function f is continuous at $-3<x<3$

Fourth case If $k=3$,

LHL = $\lim\limits_{x\to k^-} f(x)=\lim\limits_{x\to k^-}(-2x)=-2k$

RHL = $\lim\limits_{x\to k^+} f(x)=\lim\limits_{x\to k^+}(6x+2)=6k+2$,

Here, at $x=3$, LHL $\neq$ RHL.

Hence, the function f is discontinuous at $x=3$

Fifth case If, $k>3$,

$f(k)=6k+2$ and $\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k}(6x+2)=6k+2$,

Here, $\lim\limits_{x\to k} f(x)=f(k)$

Hence, the function f is continuous *for all* $x\ge 3$

Hence, the function f is discontinuous *for all* at $x=3$

Question 26: If f(x) is continous at $x = \frac{\pi}{2}$, then find the value of k

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{If } x \neq \frac{\pi}{2} \\ 3, & \text{If } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer 26:

Given that the function is continuous at $x = \frac{\pi}{2}$.

Therefore, LHL = RHL = $f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{k\cos x}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}^+} \frac{k\cos x}{\pi - 2x} = 3$$

$$\Rightarrow \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = 3$$

$$\Rightarrow \lim_{h\to 0} \frac{k\sin h}{2h} = \lim_{h\to 0} \frac{-k\sin h}{-2h} = 3$$

$$\Rightarrow \frac{k}{2} = \frac{k}{2} = 3 \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$\Rightarrow k = 6$$

Question 30:

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{If } x \leq 2 \\ ax + b, & \text{If } 2 < x < 10 \\ 21, & \text{If } x \geq 10 \end{cases}$$

is a continuous function.

Answer 30:

Given that the function is continuous at $x = 2$.

Therefore, LHL = RHL = $f(2)$

$$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} 5 = \lim_{x\to 2^+} ax + b = 5$$

$$\Rightarrow 2a + b = 5 \qquad \ldots (1)$$

Given that the function is continuous at $x = 10$.

Therefore, LHL = RHL = $f(10)$

$$\Rightarrow \lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x\to 10^-} ax + b = \lim_{x\to 10^+} 21 = 21$$

$$\Rightarrow 10a + b = 21 \qquad \ldots (2)$$

Solving the equation (1) and (2), we get

$a = 2$. $b = 1$

Question 31 :

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer 31:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h ($f = goh$). Where, $g(x) = |x|$ and $h(x) = \cos x$. If g and h both are continuous function then f also be continuous.
$[\because goh(x) = g(h(x)) = g(\cos x) = |\cos x|]$

We know that |x| and cos x both are continuous functions , therefore their composition function will also be continuous.

Question 32:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer 32:

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and h ($f = goh$). Where, $g(x) = \cos x$ and $h(x) = x^2$. If g and h both are continous function,then f also be continuous.
$[\because goh(x) = g(h(x)) = g(x^2) = \cos(x^2)$

Function $g(x) = \cos x$

Let, k be any real number. At $x = k$, $g(k) = \cos k$

$\lim_{x \to k} g(x) = \lim_{x \to k} \cos x = \lim_{h \to 0} \cos(k + h) = \lim_{h \to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, $\lim_{x \to k} g(x) = g(k)$, Hence, the function g is continuous for all real numbers.

Function $h(x) = x^2$

Let, k be any real number. At $x = k$, $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

Here, $\lim_{x \to k} h(x) = h(k)$, Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Summary

- *Definition:* $f(x)$ *is* <u>*continuous*</u> *at* $x = c$ *if* $\lim_{x \to c} f(x) = f(c)$
  - *Right-continuous at* $x = c$ *if* $\lim_{x \to c^+} f(x) = f(c)$
  - *Left-continuous at* $x = c$ *if* $\lim_{x \to c^-} f(x) = f(c)$
- *If* $f(x)$ *is continuous at all points in its domain, f is simply called continuous.*
- *There are* three *common types of discontinuities: removable, jump, infinite.*

- *A removable discontinuity can often be fixed using an extension of the original function.*
- *There are properties of continuity: sums, products, multiples, differences, quotients (when the denominators ≠ 0) and composites are also continuous.*
- *Basic functions: Polynomials, rational functions, nth-root and algebraic functions, trig functions and their inverses, exponential and log functions are continuous* ***on their domains.***
- *The Intermediate Value Theorem can be used to determine if a certain f(x) value must exist over a certain interval.*
- *If f is continuous over the range of g and g is continuous over its domain then fog is a continuous function over domain of g.*